and from (4.3) and the positivity $\rho(q^2)\ge\rho_{2\pi}(q^2)$,

$$|\mathrm{Im}F_\pi(q^2)|^2\le 6\pi^2\left(\frac{4q^2}{q^2-4\mu^2}\right)^{3/2} q^2\rho(q^2)\,. \tag{4.6}$$

If we further use the relation valid to first order in $\alpha=e^2/4\pi$, between the photon spectral function and the annihilation cross section $\rho(q^2)=\sigma_T{}^{e^+e^-}(q^2)/8\pi^2\alpha$, where q^2 is the c.m. energy of the pair, and combining (4.5) and (4.6), we have our relative bound

$$\left|\frac{F_\pi(q^2)-e}{q^2}\right|_{q^2\le 0}\le\int_{4\mu^2}^{\infty}\frac{dq'^2}{q'^2(q'^2+|q|^2)}\left(\frac{4q'^2}{q'^2-4\mu^2}\right)^{3/4}\times\left[\frac{3q'^2\sigma_T{}^{e^+e^-}(q'^2)}{\pi e^2}\right]^{1/2}. \tag{4.7}$$

There immediately follows from this expression valid for $q^2\le 0$ a bound on the charge radius of the pion:

$$\langle r_\pi{}^2\rangle=\frac{6}{F_\pi(0)}\frac{dF_\pi(q^2)}{dq^2}\bigg|_{q^2=0}\le 6\int_{4\mu^2}^{\infty}\frac{dq^2}{q^4}\left(\frac{4q^2}{q^2-4\mu^2}\right)^{3/4}\times\left[\frac{3q^2\sigma_T{}^{e^+e^-}(q^2)}{\pi e^4}\right]^{1/2}. \tag{4.8}$$

Also perhaps of interest is the observation that if $F_\pi(q^2)\to 0$, $-q^2\to\infty$, then (4.7) implies

$$\int_{4\mu^2}^{\infty}\frac{dq'^2}{q'^2}\left(\frac{4q'^2}{q'^2-4\mu^2}\right)^{3/4}\left[\frac{3q'^2\sigma_T{}^{e^+e^-}(q^2)}{\pi e^4}\right]^{1/2}\ge 1\,. \tag{4.9}$$

The integral here probably diverges as is expected in the quark-algebra estimate of $\sigma_T{}^{e^+e^-}(q^2)$, $q^2\to\infty$, so the inequality is trivial.

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Relation Between the Multi-Regge Model and the Missing-Mass Spectrum*

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The integral equation approach to the multi-Regge peripheral model is applied to give the missing-mass spectrum with Regge behavior in s and M^2. A simple factorizable model for the double Regge coupling then gives the magnitude and t dependence of the cross section. This model is found to be in reasonable agreement with the backward $\pi^-+p\to p+X^-$ data.

THE integral equation approach to the multi-Regge production model for computing the contribution to the elastic absorptive part has recently been formulated.[1–5] The approach has been used to predict total cross sections at high energies with results that are encouraging.[6] Recently, this approach has also been applied by Caneschi and Pignotti[7] to studying the missing-mass spectrum at high energies. The importance of the missing-mass experiments as a test of the integral equation approach has motivated us to examine this relationship in more explicit detail. In this paper we show how the integral equations may be readily applied to give the missing-mass spectrum in high-energy inelastic collisions in terms of the forward Reggeon-particle absorptive amplitude $\mathcal{A}(t;s)$. The Reggeon-particle absorptive amplitude at general momentum transfer, forward or nonforward, can be obtained by solving a multi-Regge integral equation.[3–5] The predicted missing-mass spectrum has characteristic properties which can be tested experimentally. Furthermore, using the simplified model of a factorizable and ω-angle-independent double Regge coupling, we achieve an expression of the missing-mass cross section entirely in terms of two-body cross sections and coupling constants. This allows us to predict not only the Regge behavior in energy and missing mass, but also the magnitude of the missing-mass cross section. The result is applied with reasonable agreement to "backward" $\pi^-+p\to p+X^-$ reaction,[8] as production on the end of a multiperipheral chain. In a later paper,[9] we will examine more explicitly the formulation and results of the missing-mass contribution from the particles emitted from the central

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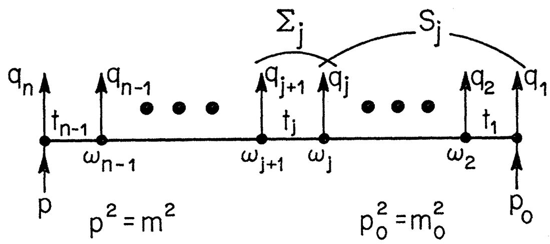

FIG. 1. Kinematics of multi-Regge chain.

region of a multiperipheral chain. This work emphasizes the importance of missing-mass experiments to the search for a "realistic" multiperipheral model.

We begin by demonstrating how the missing-mass spectrum of a scattering process is obtained from the forward Reggeon-particle absorptive amplitude $\mathcal{A}(t;s)$, where t is the square of the mass of the Reggeon. For simplicity, let us consider a multi-Regge model for a spinless particle. The production amplitude for a 2 to n process (see Fig. 1) depends on the invariants

$$\Sigma_j=(q_{j+1}+q_j)^2,\quad t_j=(p_0-\sum_{i=1}^{j} q_i)^2,\quad j=1,2,\ldots,n-1,$$

and the Toller angles ω_j, $j=2,3,\ldots,n-1$.[10] This amplitude is approximated in the multi-Regge region by the product form

$$T_n=G_lH_{n-1}\beta_{n-1}H_{n-2}\beta_{n-2}\cdots\beta_2H_1G_r. \tag{1}$$

Here $G_l(t_{n-1})$ and $G_r(t_1)$ are single Regge couplings, $\beta_j(t_j,t_{j-1},\omega_j)$ are double Regge couplings, and $H_j(t_j,\Sigma_j)$ is the Regge factor $\xi(t_j)(\Sigma_j/\mu^2)^{\alpha(t_j)}$, where $\xi(t)$ contains the signature factor, physical poles in t_j, and is normalized to $1/(t-m^2)$ at the pole at $\alpha(m^2)=0$. Using the invariant energies

$$s_j\equiv(\sum_{i=1}^{j} q_i)^2$$

in the multi-Regge region of large s_j and Σ_j, and using the result of strong ordering[11] that $s_{j+1}\gg s_j$, we have the kinematic relationship

$$\Sigma_j\approx\frac{s_{j+1}}{s_j}f(t_j,t_{j-1};\omega_j), \tag{2}$$

where[3–5]

$$f(t_j,t_{j-1};\omega_j)=\frac{\Delta(t_j,t_{j-1},\mu^2)}{\mu^2-t_j-t_{j-1}+2(t_jt_{j-1})^{1/2}\cos\omega_j}, \tag{3}$$

with

$$\Delta(x,y,z)=x^2+y^2+z^2-2xy-2yz-2zx.$$

For simplicity, we will use the form (1) for the production amplitude both in the high-energy and the low-energy regions.

We will now derive the missing-mass spectrum for the observed particle emerging from the left end of Fig. 1 and then present the general result for the observed particle emerging from any position in the multi-Regge chain. The contribution of the n-particle intermediate state to the two-body absorptive part is

$$A_n(s)=\tfrac{1}{2}\int d\Phi^n|T_n|^2, \tag{4}$$

where the n-particle phase space is

$$d\Phi^n=(2\pi)^4\delta^4(\sum_{j=1}^{n} q_j-p-p_0)\prod_{k=1}^{n}\frac{d^4q_k}{(2\pi)^3}\delta^+(q_k^2-\mu^2). \tag{5}$$

The two-body absorptive part is given by

$$A(s)=\sum_{n=1}^{\infty}A_n(s),$$

and we can relate this to the Reggeon-particle absorptive part $\mathcal{A}(t;s)$ [3–5] by removing the end vertex and Regge factor from $[A(s)-A_1(s)]$, $A_1(s)=\pi\delta(s-\mu^2)\times G_r^2(\mu^2)$, i.e.,

$$A(s)-A_1(s)=\int\frac{d^4q}{(2\pi)^3}\delta^+(q^2-\mu^2)G_l(t)^2\xi(t)^2\times\left(\frac{s}{M^2}\right)^{2\alpha(t)}\mathcal{A}(t;M^2). \tag{6}$$

Here q is the momentum of the left-end particle and $M^2\equiv(p+p_0-q)^2$, $t=(p-q)^2$. $\mathcal{A}(t;M^2)$ contains a single-particle-state contribution $\pi\delta(M^2-\mu^2)G_r(t)^2$. The integral equation for $\mathcal{A}(t;M^2)$ has been previously derived and studied.[3–5] At large M^2, it has been shown to have Regge behavior $\mathcal{A}(t;M^2)\propto(M^2/\mu^2)^{\alpha(0)}$. The total cross section is related to the two-body absorptive part by

$$\sigma_{\text{tot}}(s)=\frac{1}{2\Delta^{1/2}(s,m^2,m_0^2)}\sum_{n=2}^{\infty}\int d\Phi^n|T_n|^2=\frac{A(s)-A_1(s)}{\Delta^{1/2}(s,m^2,m_0^2)}. \tag{7}$$

To obtain the cross section for missing mass M^2, observing the final particle of momentum q, we undo the phase-space integral over q in Eqs. (7). Using Eq. (6), we obtain

$$\frac{d\sigma}{d^4q\,\delta^+(q^2-\mu^2)}=\frac{1}{(2\pi)^3}\frac{1}{\Delta^{1/2}(s,m^2,m_0^2)}G_l(t)^2\xi(t)^2\times\left(\frac{s}{M^2}\right)^{2\alpha(t)}\mathcal{A}(t;M^2). \tag{8}$$

[10] N. F. Bali, G. F. Chew, and A. Pignotti, Phys. Rev. 163, 1572 (1967).

[11] F. Zachariasen and G. Zweig, Phys. Rev. 160, 1322 (1967).

The inclusion of the cases where the detected particle comes from any position on the chain can best be expressed in terms of the auxiliary function[12]

$$B(p,\, p-q;\, p_0)=\xi^2((p-q)^2)\left[\frac{(p+p_0)^2}{(p-q+p_0)^2}\right]^{2\alpha((p-q)^2)}\times\mathfrak{A}((p-q)^2;\,(p-q+p_0)^2)\,. \quad (9)$$

Finally, the total contribution to the missing-mass spectrum[9] is given by (see Fig. 2)

$$\frac{d\sigma}{d^4q\,\delta^+(q^2-\mu^2)}=\frac{1}{(2\pi)^3\Delta^{1/2}(s,m^2,m_0^2)}\times\Big[G^2((p-q)^2)B(p,\,p-q;\,p_0)+\frac{2}{(2\pi)^4}\int d^4p'\int d^4p_0'\times\delta^4(p'+p_0'+q)B(-p_0',\,p';\,p)\beta(p'^2,p_0'^2,\omega)^2\times B(-p',\,p_0';\,p_0)+G^2((p_0-q)^2)B(p_0,\,p_0-q;\,p)\Big]. \quad (10)$$

Viewing the collision in the c.m. system, these three terms describe, respectively, the spectrum of the observed particle for large positive, intermediate, and large negative longitudinal momentum.[7,9]

Through Eq. (10) and the Regge behavior of $\mathfrak{A}(t;\,M^2)$, we now have enough information to perform multi-Regge fits to missing-mass data.[7] We can also learn much more if we can find the magnitudes of the multi-Regge graphs from our knowledge of the magnitudes of Reggeized two-body elastic scattering and total cross sections. In order to do this we consider the simple model of an ω-angle-independent[13] and factorizable internal Regge residue $\beta(t_1,t_2)$. It is normalized for one particle on the mass shell to be the single Regge coupling $\beta(t_1,m^2)=G(t_1)$, $\beta(m^2,t_2)=G(t_2)$, where $G(m^2)=g$, so that the factorizable residue is

$$\beta(t_1,t_2)=G(t_1)G(t_2)/g\,. \quad (11)$$

For small t's the integral equation is ω independent and becomes

$$\mathfrak{A}(t;\,s)=\pi\delta(s-\mu^2)G_r(t)^2+\int\frac{d^4p'}{(2\pi)^3}\delta(p'^2-\mu^2)\xi(t')^2\frac{G(t)^2G(t')^2}{g^2}\left(\frac{s}{s'}\right)^{2\alpha(t')}\times\mathfrak{A}(t';\,s')\,. \quad (12)$$

[12] The $\mathfrak{A}$ absorptive part used in our formulation may also be considered as an approximation of the CGL (Ref. 1) equation for B in the strong ordered kinematic region of Eq. (2). B will then be related to $\mathfrak{A}$ by Eq. (9), as shown in Refs. 3–5. The general result (10) is valid as well for the B function of CGL without this kinematic approximation.

[13] C.-I Tan and J. M. Wang, Phys. Rev. **185**, 1899 (1969).

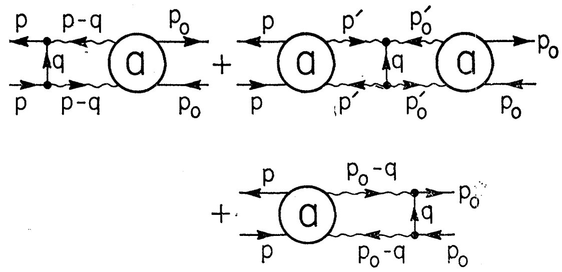

FIG. 2. Multi-Regge diagrams for the missing-mass spectrum in terms of the Reggeon-particle absorptive amplitude $\mathfrak{A}(t;\,M^2)$.

Comparing the integral with Eq. (6), we find

$$\mathfrak{A}(t;\,s)=[G(t)^2/g^2]A(s)\,, \quad (13)$$

where we use the approximation that the t dependence from the integration limits is small.[3–5] Because of Eq. (7) for $A(s)$, we have now a model for $\mathfrak{A}(t;\,s)$ in terms of known two-body total cross sections.

For illustration of its usefulness, we apply Eq. (13) to Eq. (8), observing that the elastic differential cross section at s-channel invariant $(s\mu^2/M^2)$ is

$$\frac{d\sigma}{dt}\left(\frac{s}{M^2}\mu^2,t\right)=\frac{1}{16\pi\Delta((s/M^2)\mu^2,m^2,m_0^2)}\times\left|G(t)\xi(t)\left(\frac{s}{M^2}\right)^{\alpha(t)}G(t)\right|^2. \quad (14)$$

This gives the missing-mass spectrum in terms of known two-body Regge processes,

$$\frac{d\sigma}{d^4q\,\delta^+(q^2-\mu^2)}=\frac{2}{\pi}\Delta^{1/2}(s,m^2,m_0^2)\frac{d\sigma}{dt}(s,t)\delta(M^2-\mu^2)+\frac{2}{\pi^2}\frac{\Delta(s\mu^2/M^2,m^2,m_0^2)}{\Delta^{1/2}(s,m^2,m_0^2)}\frac{d\sigma}{dt}\left(\frac{s}{M^2}\mu^2,t\right)\frac{1}{g^2}\times\Delta^{1/2}(M^2,m^2,m_0^2)\sigma_{\rm tot}(M^2)\,, \quad (15)$$

where we have isolated the contribution from elastic scattering. It is interesting to note that the t dependence of the end term contribution is that of elastic scattering.

In most cases, the existence of several communicating Regge trajectories will complicate the situation. To illustrate our approach, therefore, we have chosen a particular production reaction where only one known trajectory can be exchanged. We consider the reaction $\pi^-+p\rightarrow p+X^-$,[8] where the outgoing proton has small momentum transfer to the incoming pion. If we further restrict ourselves to those events where the outgoing particle has large longitudinal momentum (for 16 GeV/c π^- incident, we detect only $|\mathbf{q}|_{\rm lab}$ of proton ≥ 10 GeV/c, or $M^2<7$ GeV^2) two major simplifications occur: (i) Among the three terms in Eq. (10), only the "left-end" term can contribute substantially, and (ii) because of the charge configuration, the only Regge

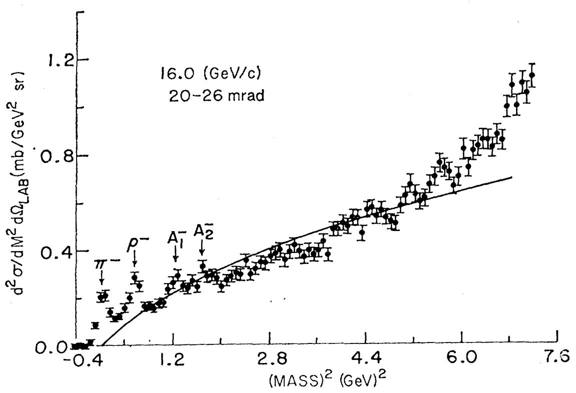

FIG. 3. Missing-mass spectrum for $\pi^- p \to pX^-$. The solid line is the fit from the multi-Regge factorizable model Eq. (16).

trajectory that can be exchanged is the Δ trajectory. (The ω-angle-independent model can be formulated for an exchanged particle with spin by assuming that only one helicity amplitude is dominant.[14]) Consequently, we obtain, for $\pi^- + p \to p + X^-$ in the laboratory frame,

$$\frac{d\sigma}{d\Omega dM^2}=\phi\times 16\pi\Delta\left(\frac{s}{M^2}\mathrm{GeV}^2, m_\pi^2, m_N^2\right)\times\frac{d\sigma^{\pi^- p}}{du}\left(\frac{s}{M^2}\mathrm{GeV}^2, t\right)\frac{A_{\bar{\Delta}p}(M^2)}{g_{N\pi\bar{\Delta}}^2}, \qquad (16)$$

$$\phi\simeq|\bar{q}|/8(2\pi)^3 M_N^2|\bar{p}_\pi|,$$

where $A_{\bar{\Delta}p}(M^2)$ is the absorptive part of physical $\bar{\Delta}p$ scattering amplitude and $g_{N\pi\bar{\Delta}}^2$ is the coupling constant.

[14] T. W. B. Kibble, Phys. Rev. 131, 2282 (1963).

This prediction is compared with the experimental result of Anderson *et al.*,[8] and the fit to the experiment is shown in Fig. 3. We have parametrized the data for $d\sigma/du$ for backward[15] $\pi^- p$ scattering with $\alpha_\Delta(t)=0.049+0.76t$, and also used $g_{\pi N\bar{\Delta}}^2=30$ GeV^2. The absorptive part is given by a sum of a Pomeranchon term and an "average" meson term of the form $M^2[\sigma_P+\sigma_M(M^2/1\ \mathrm{GeV}^2)^{\alpha_M(0)-1}]$ [we have used $\alpha_P(0)=1$ and $\alpha_M(0)\simeq 0.7$]. In analogy to the $\bar{p}p$ total cross section, we use $\sigma_P\simeq 45$ mb and $\sigma_M\simeq 120$ mb, although the fit is not sensitive to this particular choice.[16]

We have shown that the multi-Regge integral equation is convenient for describing the missing-mass spectrum in addition to its previous use for the two-body absorptive part (total cross section). The additional observables in the missing-mass spectrum provide a more demanding test, since Regge behavior is required in M^2 as well as in s. This test has so far been met in the experiment analyzed above and in other experiments.[7] Using a simple factorizable model for the double Regge residue, we find that the integral equations can be used as a practical method to calculate the magnitudes of the amplitudes, in reasonable agreement with experiment. Further missing-mass experiments should prove most useful toward constructing and testing more realistic multiperipheral models.

We wish to acknowledge the stimulating remarks and encouragement of Professor M. L. Goldberger. One of us (C.I.T.) wishes to thank the Aspen Center for Physics for their hospitality.

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[16] The discrepancy at large M^2 seems to indicate the gradual importance of the "central" diagram of Eq. (10).

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$\gamma\pi \to \pi\pi$ in the Veneziano Model and the Lifetime of the Neutral Pion

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The amplitude for the process $\gamma\pi \to \pi\pi$ given in terms of a five-point generalized Veneziano function is shown to remove an inconsistency in the application of the Veneziano method to the process $\pi+N \to \pi+N+\gamma$ in which the first process is known to dominate. A determination of the relevant residue function consistent with other information leads to a value of the neutral pion lifetime in good agreement with experiment, provided use is made of the idea of effective width of the ρ meson.

THE difficulty of writing down the Veneziano amplitude for the process

$$\gamma(p_1)+\pi_\alpha(p_2) \to \pi_\beta(p_3)+\pi_\gamma(p_4) \qquad (1)$$

has been pointed out by several authors.[1,2] The matrix element of the reaction is written in the form

$$M_{fi}=\epsilon_{\alpha\beta\gamma}\epsilon^{\mu\nu\lambda\sigma}\epsilon_\mu p_{2\nu}p_{3\lambda}p_{4\sigma}A(s,t,u), \qquad (2)$$

where the Mandelstam variables s, t, and u are connected by the relation $s+t+u=3m_\pi^2$. Since only isoscalar photons can contribute, the process is identical to the one originally considered by Veneziano, i.e., $\omega\pi \to \pi\pi$, provided m_ω is taken to be zero. But in order to eliminate undesirable poles with even angular momenta, Veneziano required in his process the con-

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[2] I. Raszillier and D. H. Schiller, Academia Republicu Socialiste Romania Report, Institutul De Fizica, Bucharest, 1969 (unpublished).